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ALGORITHMS FOR FAST RECURSIVE ESTIMATION IN
LINEAR SYSTEMS WITH CONSTANT PARAMETERS
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CHANDRASEKHAR-TYPE ALGORITHMS
FOR FAST RECURSIVE ESTIMATION IN
LINEAR SYSTEMS WITH CONSTANT PARAMETERS.

by

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ABSTRACT

Recursive least-squares estimates for processes that can be generated from finite-dimensional linear systems are usually obtained via an $n \times n$ matrix Riccati differential equation, where n is the dimension of the state space. In this new recursive method the gain matrix for the Kalman filter and the covariance of the state vector are computed not via the Riccati equation, but from certain other equations. These differential equations are said to be of Chandrasekhar-type, because they are similar to certain equations introduced in 1948 by the astrophysicist, S. Chandrasekhar, to solve finite-interval Wiener-Hopf equations arising in radiative transfer. Chandrasekhar extended Ambarzumian's invariance principles to solve the above problem and in fact in recent years this served as a stimulus for much activity by Bellman in the development of an invariant approach to the solution of various transport processes. The "invariant imbedding" idea resulted in the reduction of the basic boundary value problem of transport theory to an equivalent initial value system, a significant computational advance.

Initial experience has shown that there is some computational savings in the new method and the loss of positive definiteness of the covariance matrix is less vulnerable.

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CHAPTER I

INTRODUCTION

The work of R.E. Kalman is a most significant contribution to filtering and prediction theory, since the original work of Weiner (8). The Kalman filtering theory introduces a new look at the classical problems of prediction, smoothing, and filtering. More specifically, Kalman's method has the following features:

1. The linear dynamic system is described by the state variables and state equations. This not only represents a modern approach to the systems problem, but also makes machine computation simpler.
2. The Kalman filtering theory treats stationary and nonstationary random processes, single-variable and multivariable systems, all in a unified manner.

The Kalman filtering problem can be stated in general as: given $y(t) = Z(t) + v(t)$, where $y(t)$ is a message in the form of a signal corrupted by additive noise and $Z(t)$ is the actual signal and $v(t)$ is the noise, determine the value of the $Z(t)$ in the sense of the minimum mean-square error. The error is defined to be the difference between the actual output of the filter and the signal component of the input message.

Generally to determine the value of $Z(t)$ at sometime $t = t_j$, given the measured or observed value $\{y(\tau), t_0 \leq \tau \leq t_k\}$, the time t_j can be less than, equal to, or greater than t_k . These three cases

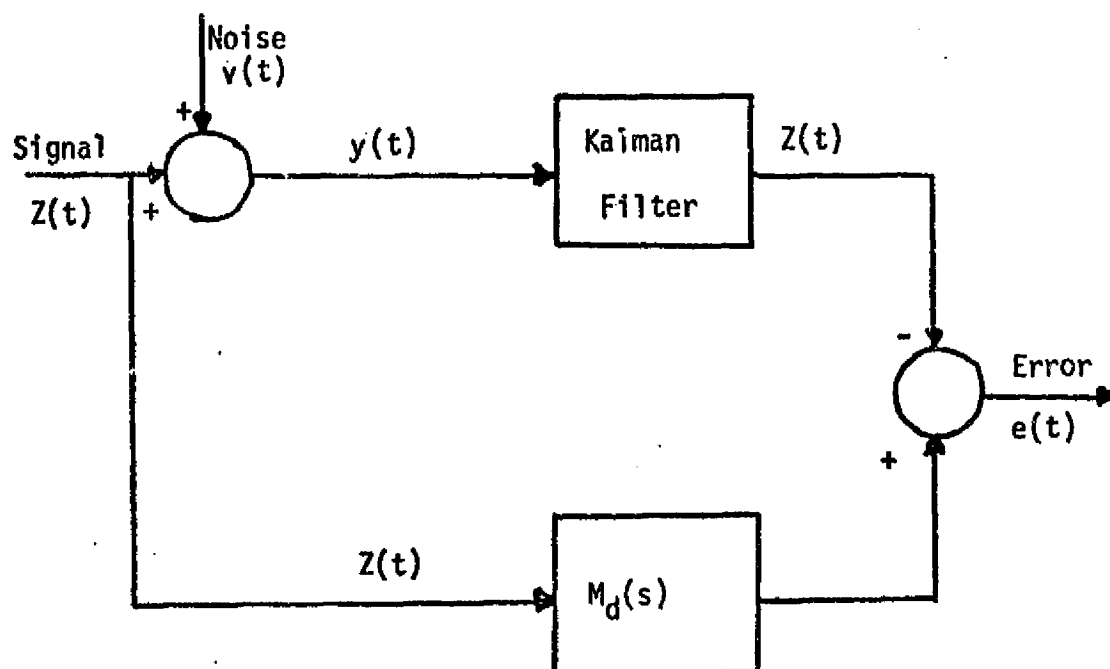
can be defined as

$t_j < t_k$ smoothing (interpolation)

$t_j = t_k$ filtering

$t_j > t_k$ predicting

The block diagram illustrating the general philosophy of the Kalman filter is shown in Fig. 1.



$M_d(s) = e^{-\tau s} \quad (\tau > 0) \quad \text{Smoothing (Interpolation)}$

$M_d(s) = 1 \quad \text{Filtering}$

$M_d(s) = e^{\tau s} \quad (\tau > 0) \quad \text{Predicting}$

FIG. 1 KALMAN FILTER

This research will include the study and comparison of a new algorithm for recursive state estimation via Chandrasekhar-type equations (5).

By this method, the gain matrix for the Kalman filter is obtained directly, without having to solve separately for the error-covariance matrix. In general, the gain matrix is obtained by solution of $n(n+1)/2$ simultaneous nonlinear differential equation of Riccati-type, where n is the dimension of the state space. But in this new method, it only requires the solution of $n(m+p)$ simultaneous nonlinear differential equations, where m and p are the dimension of the input and observation processes, respectively. In most practical cases $n > p + m$, and our experience shows that whenever $n > p + m$, there is some computational saving in using this new method.

Notation

The "dot" notation will be used for derivatives (i.e., $\frac{dx}{dt} \equiv \dot{x}$);
and "prime" will be used for transpose of a matrix as shown (F').

CHAPTER II

FORMULATION OF PROBLEM AND SOLUTION TECHNIQUE

Problem Statement

Consider the standard Kalman state-space model where the problem is to calculate linear least-square estimates of a signal process $z(\cdot)$ from p -vector observations of the form

$$y(t) = Z(t) + v(t) \quad t \geq t_0 \quad (2-1)$$

where

$$E[v(t)v'(s)] = I_p \delta(t - s) \quad (2-2)$$

and $z(\cdot)$ is given by a state-space model

$$Z(t) = HX(t), \quad t \geq t_0 \quad (2-3)$$

$$\dot{X}(t) = FX(t) + GU(t), \quad X(t_0) = X_0 \quad (2-4)$$

where $x(\cdot)$ is an n -vector, $U(\cdot)$ is a m -vector, and

$$E[X_0] = 0, \quad E[X_0 X_0'] = \Pi_0, \quad E[U(t) X_0'] = 0 \quad (2-5)$$

The input process $u(\cdot)$ is white with covariance function

$$E[U(t)U'(s)] = Q\delta(t - s), \quad E[U(t) v'(s)] = C\delta(t - s) \quad (2-6)$$

The matrices F , G , H , Π_0 , Q , C are assumed to be known and for simplicity we also assume that

$$H \text{ has full rank } p \quad (2-7)$$

It is desired to calculate

$\hat{x}(t)$ = the linear least-square estimate of $x(t)$, given $\{y(\tau), t_0 \leq \tau < t\}$.

By linearity it follows that

$$\hat{z}(t) = H\hat{x}(t) \quad (2-8)$$

The Kalman filter for this problem is

$$\dot{\hat{x}}(t) = F\hat{x}(t) + K(t)(y(t) - H\hat{x}(t)), \quad \hat{x}(t_0) = 0 \quad (2-9)$$

where

$$K(t) = P(t)H' + GC \quad (2-10)$$

The form of this best linear estimator, which is called the Kalman filter is shown in Fig. 2. (7)

The $n \times n$ matrix $P(\cdot)$ is the mean-square error in the estimated of the states

$$P(t) = E[x(t) - \hat{x}(t)][x(t) - \hat{x}(t)]' \quad (2-11)$$

This matrix is computed as the unique solution of the non-linear matrix differential equation of Riccati type.

$$\begin{aligned} \dot{P}(t) &= FP(t) + P(t)F' - K(t)K'(t) + GQG', \\ P(t_0) &= \Pi_0 \end{aligned} \quad (2-12)$$

since $P(\cdot)$ is symmetric, this Riccati equation involves the solution of $n(n+1)/2$ simultaneous nonlinear differential equations, which has generally to be carried out on an analog or digital computer. The fact, clear from (2-11), that $P(\cdot)$ has to be nonnegative-definite often places certain stringent accuracy requirements on the computation and may require special attention. However, by this method $K(\cdot)$ is

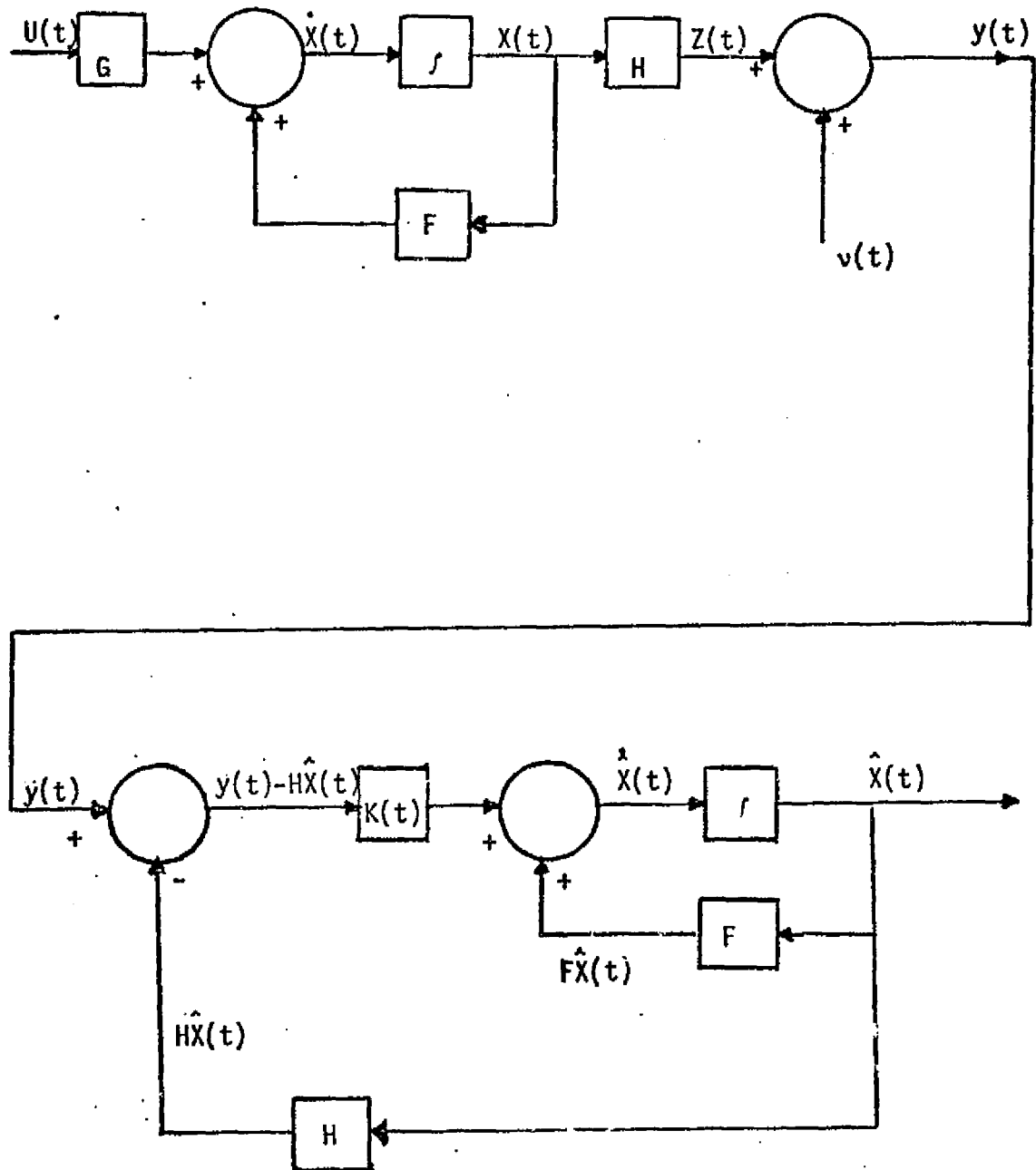


Fig. 2. The continuous-time Kalman-filter

calculated not via Riccati equation but from certain equations, which are called of Chandrasekhar-type.

Chandrasekhar-Type Equations

The Chandrasekhar-type equations are as follow (5):

$$\dot{K}(t) = [Y_1(t)Y_1'(t) - Y_2(t)Y_2'(t)]H', \quad K(t_0) = \pi_0 H' + GC \quad (2-13)$$

$$\dot{Y}_i(t) = [F - K(t)H]Y_i(t), \quad Y_i(t_0) = L_i, \quad i = 1, 2 \quad (2-14)$$

The initial condition matrices L_i are to be determined from the following procedure

$$D \triangleq F\pi_0 + \pi_0 F' + GQG' - (\pi_0 H' + GC)(\pi_0 H' + GC)' \quad (2-15)$$

Let us assume that D has rank α , and $\alpha \leq n$. From (2-15) it is clear that D is a symmetric matrix and it will have real eigenvalues. Here in various ways, which will be described later, we can write D as

$$D = [L_1 L_1' - L_2 L_2'] \quad (2-16)$$

If we assume that

β = the number of positive eigenvalues of D .

Then the dimensions of matrices L_i would be

$$L_1 = n \times \beta, \quad L_2 = n \times (\alpha - \beta) \quad (2-17)$$

where the number of positive eigenvalues of D is to be calculated, however, there is no need to calculate the eigenvalues.

The best way to find the initial conditions L_i is to use Cholesky Decomposition which will be described later in this chapter.

Therefore let us make a so-called lower triangular-diagonal-upper triangle (LDU) decomposition of D as

$$D = L_0 S L_0' = [L_1 \ L_2] S [L_1' \ L_2'] \quad (2-18)$$

where we define S to be the $\alpha \times \alpha$ signature matrix of D

$$S = \text{diag}\{1, 1, \dots, 1, -1, \dots, -1\}$$

Now by these assumptions and with (2-18) we can write the Chandrasekhar-type equations more compactly as

$$\dot{K}(t) = Y(t) S Y'(t) H', \quad K(t_0) = \Pi_0 H' + G C \quad (2-19)$$

$$\dot{Y}(t) = [F - K(t)H] Y(t), \quad Y(t_0) = [L_1 \ L_2] \quad (2-20)$$

where

$K(\cdot)$ is an $(n \times p)$ matrix

$Y(\cdot)$ is an $(n \times \alpha)$ matrix

F is an $(n \times n)$ matrix

H is an $(p \times n)$ matrix

The important aspect of these two last equations (2-19), (2-20) is that the Kalman gain $K(\cdot)$ can be found directly without going through the matrix $P(\cdot)$. Note that if it is desired to find $P(\cdot)$, we may use (2-10) to write

$$\dot{P}(t) = Y(t) S Y'(t), \quad P(t_0) = \Pi_0 \quad (2-21)$$

We would like to develop the new algorithm based on the Chandrasekhar-type equations (2-13) and (2-14). Here we have $n(\alpha + p)$ simultaneous nonlinear equations. In many practical cases α is less than or equal to p or to m , where m is the number of inputs; and often

$m = p = 1$.

Comparing $n(n+1)/2$ simultaneous nonlinear differential equations of Riccati-type with $n(\alpha+p)$ simultaneous nonlinear differential equations of Chandrasekhar-type, we see that whenever $n \geq p + \alpha$, we will have a substantial computational saving in using the Chandrasekhar-type equations (2-13) and 2-14). Good experience which has been obtained in several examples, shows that the equations (2-13) and (2-14) are numerically well behaved.

Let us now define the steady state behavior ($t \rightarrow \infty$) of $K(\cdot)$ as follow

$$\bar{K} = \lim_{t \rightarrow \infty} K(t) \quad (2-22)$$

Because (2-13) and (2-14) are a set of simultaneous equations, it is computationally preferable to find \bar{K} by computing the solution of Chandrasekhar-type equation until the solution remains fairly constant. This will be shown on an example later in this chapter.

Determination of the Initial Condition Matrices L_1 and L_2

CHOLESKY DECOMPOSITION

Any symmetric positive semidefinite $n \times n$ matrix D may be written in the factored form (6):

$$\begin{bmatrix} D_{11} & D_{12} & \dots & D_{1n} \\ D_{12} & D_{22} & & \\ \cdot & & \cdot & \\ \cdot & & \cdot & \\ \cdot & & \cdot & \\ D_{1n} & \dots & D_{nn} \end{bmatrix} = \begin{bmatrix} L_{11} & 0 & \dots & 0 \\ L_{21} & L_{22} & \dots & 0 \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & 0 \\ L_{n1} & \dots & L_{nn} \end{bmatrix} \begin{bmatrix} L_{11} & L_{21} & \dots & L_{ni} \\ 0 & L_{22} & & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ 0 & 0 & \dots & L_{nn} \end{bmatrix}$$

This is called lower triangle-upper triangular decomposition. Cholesky gave the following recursive algorithm for computing L .

For $i = 1, n$,

$$L_{ii} = \sqrt{D_{ii} - \sum_{j=1}^{i-1} L_{ij}^2}$$

$$L_{ji} = \begin{cases} 0, & j < i \\ \frac{1}{L_{ii}} (D_{ji} - \sum_{k=1}^{j-1} L_{jk} L_{ik}), & j = i+1, n \end{cases}$$

Example: 1

$$D = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 2 \\ 3 & 2 & 14 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & \sqrt{8-4} & 0 \\ 3 & (2-8/2) & \sqrt{14-9-4} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & -2 & 1 \end{bmatrix}$$

Lower Triangular-diagonal-Upper Triangular Decomposition
(LDU)

$$D = L S L'$$

$$S = \begin{bmatrix} 1 & 0 & \dots & 0 \\ & 1 & & \\ 0 & \beta & & \\ & & \ddots & \\ & & & -1 \\ 0 & & & \alpha & -1 \end{bmatrix}$$

Example: 2

$$D = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 2 \\ 3 & 2 & 12 \end{bmatrix}$$

where $\alpha = 3$, $\beta = 2$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 2 \\ 3 & 2 & 12 \end{bmatrix} = \begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} L_{11} & L_{21} & L_{31} \\ 0 & L_{22} & L_{32} \\ 0 & 0 & L_{33} \end{bmatrix}$$

After multiplying the right-hand side and equating the corresponding elements

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & -2 & 1 \end{bmatrix}$$

Fortran algorithm for $D = LSL^T$

Consider:

$$\begin{bmatrix} D_{11} & D_{21} \dots & D_{1n} \\ D_{21} & D_{22} \dots & D_{2n} \\ \vdots & \vdots & \vdots \\ D_{n1} & D_{n2} & D_{nn} \end{bmatrix} = \begin{bmatrix} L_{11} & & 0 \\ L_{21} & L_{22} & \\ \vdots & \vdots & \vdots \\ L_{n1} & L_{n2} & L_{nn} \end{bmatrix} \begin{bmatrix} S_1 & & 0 \\ & S_2 & \\ & & \ddots \\ 0 & & S_n \end{bmatrix} \begin{bmatrix} L_{11} & L_{21} \dots & L_{n1} \\ & L_{22} & L_{n2} \\ & & \ddots & L_{nn} \end{bmatrix}$$

Form Column 1 of L

$$L_{11} = \sqrt{D_{11}/S_1}$$

$$L_{21} = D_{21}/S_1 L_{11}$$

$$L_{31} = D_{31}/S_1 L_{11}$$

\vdots

Form Column 2 of L

$$L_{22} = \sqrt{\frac{1}{S_2} (D_{22} - S_1 L_{21}^2)}$$

$$L_{32} = (D_{32} - S_1 L_{21} L_{31}) / S_2 L_{22}$$

$$L_{42} = (D_{42} - S_1 L_{21} L_{41}) / S_2 L_{22}$$

$$L_{52} = (D_{52} - S_1 L_{21} L_{51}) / S_2 L_{22}$$

Form Column 3 of L

$$L_{33} = \sqrt{\frac{1}{S_3} (D_{33} - S_1 L_{31}^2 - S_2 L_{32}^2)}$$

$$L_{43} = (D_{43} - S_1 L_{31} L_{41} - S_2 L_{32} L_{42}) / S_3 L_{33}$$

$$L_{53} = (D_{53} - S_1 L_{31} L_{51} - S_2 L_{32} L_{52}) / S_3 L_{33}$$

Form Column 4 of L

$$L_{44} = \sqrt{\frac{1}{S_4} (D_{44} - S_1 L_{41}^2 - S_2 L_{42}^2 - S_3 L_{43}^2)}$$

$$L_{54} = (D_{54} - S_1 L_{41} L_{51} - S_2 L_{42} L_{52} - S_3 L_{43} L_{53}) / S_4 L_{44}$$

$$L_{64} = (D_{64} - S_1 L_{41} L_{61} - S_2 L_{42} L_{62} - S_3 L_{43} L_{63}) / S_4 L_{44}$$

⋮

The fortran algorithm is as follows:

```

L(1,1) = SQRT(ABS(D(1,1)/S(1)))
DO 100 I=2,N
  L(1,I)=0
  IF(L(1,1).EQ.0.)L(1,1)=0
  IF(L(1,1).EQ.0.) GO TO 100
  L(1,I)=D(I,1)/(S(1)*L(1,1))
100 CONTINUE
DO 500 J=2, N
  JM1=J-1
  JP1=J+1
  L(J,J)=D(J,J)
  DO 200 K=1, JM1
200 L(J,J)=(L(J,J)-S(K)*L(J,K) **2)
  L(J,J)=SQRT(ABS(L(J,J)/S(J)))
  IF(J.EQ.N) GO TO 500
  DO 400 I=JP1,N
  L(J,I)=0.
  IF(L(J,J).EQ.0.)L(J,J)=0
  IF(L(J,J).EQ.0)L(J,J)=0.
  IF(L(J,J).EQ.0)GO TO 400
  L(I,J)=D(I,J)
  DO 300 K=1, JM1

```

300 $L(I,J)=(L(I,J)-S(K)*L(J,K)+L(I,L))/S(J)$

$L(I,J)=L(I,J)/L(J,J)$

400 CONTINUE

500 CONTINUE

We now turn our attention to investigate the variety of special cases. These special cases are of great important.

Case I: (Low Initial Uncertainty)

Let us now assume that the initial state $x(t_0)$ is fairly known, that is we may write

$$E[x_0 \hat{x}_0] = \Pi_0 = 0 \quad (2-23)$$

Substitute $\Pi_0 = 0$ into the equation (2-15), then the matrix D simplifies to

$$D = G(Q - CC')G' \geq 0 \quad (2-24)$$

Here D can have no negative eigenvalues, so that D may be written as

$$D = L_1 L_1', \quad L = n \times \alpha \quad (2-25)$$

The Chandrasekhar-type equations now can be simplified to the fewer than $n(m+p)$ equations

$$\dot{K}(t) = Y_1(t)Y_1'(t)H', \quad K(t_0) = GC \quad (2-26)$$

$$\dot{Y}_1(t) = [F - K(t)H]Y_1(t), \quad Y_1(t_0) = L_1 \quad (2-27)$$

If it is desired to find $P(\cdot)$, the mean-square error in the estimate of the states, it can be found via

$$\dot{P}(t) = Y_1(t)Y_1'(t); \quad P(t_0) = 0 \quad (2-28)$$

The above equation can also be written as

$$P(t) = \int_0^t Y_1(\tau) Y_1'(\tau) d\tau \quad (2-29)$$

A significant aspect of this last formula is that no matter how inaccurately $Y_1(\cdot)$ may have been computed, the product $Y_1(\cdot)Y_1'(\cdot)$ is always nonnegative-definite. This property may be lost in the subsequent quadrature (2-29), but we realize that it is much easier to be careful in a simple quadrature than in the solution of the Riccati-type differential equation (2-12).

Let us now consider the following example where the Kalman gain $K(\cdot)$ is found via Chandrasekhar-type equations.

Example: 3

Consider the Chandrasekhar-type equations

$$\dot{K}(t) = Y_1(t) Y_1'(t) H', \quad K(t) = GC$$

$$\dot{Y}_1(t) = [F - K(t) H] Y_1(t) \quad Y_1(t) = L_1$$

We would like to plot the values of $K(\cdot) = P(\cdot)H'R^{-1}$ for a five-state system with $\Pi_0 = 0$ and

$$F = \begin{bmatrix} -0.0297 & -1 & 0 & 0.0438 & 0 \\ 0.331 & -0.0042 & -0.0461 & 0 & 0 \\ -1.13 & 0.128 & -0.803 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$G = \begin{bmatrix} -0.0297 & 0 & 0 \\ 0.331 & 0.381 & 0.040 \\ -1.13 & 0.067 & 1.59 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$H = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad Q = \begin{bmatrix} 0.01 & 0 & 0 \\ 0 & 0.001 & 0 \\ 0 & 0 & 0.001 \end{bmatrix}$$

and with $E\{v(t)v'(s)\} = R \delta(t - s)$ where

$$R = \begin{bmatrix} 0.001 & 0 \\ 0 & 0.001 \end{bmatrix}$$

And

$$C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad L_1 = \begin{bmatrix} 0.003 \\ -0.035 \\ 0.124 \\ 0.00 \\ 0.00 \end{bmatrix}$$

Substitute the values into Chandrasekhar-type equations yield:

$$0.001 \begin{bmatrix} \dot{K}_{11} & \dot{K}_{12} \\ \dot{K}_{21} & \dot{K}_{22} \\ \dot{K}_{31} & \dot{K}_{32} \\ \dot{K}_{41} & \dot{K}_{42} \\ \dot{K}_{51} & \dot{K}_{52} \end{bmatrix} = \begin{bmatrix} Y_{11}Y_{41} & Y_{11}Y_{51} \\ Y_{21}Y_{41} & Y_{21}Y_{51} \\ Y_{31}Y_{41} & Y_{31}Y_{51} \\ Y_{41}Y_{41} & Y_{41}Y_{51} \\ Y_{51}Y_{41} & Y_{51}Y_{51} \end{bmatrix}$$

$$\begin{bmatrix} \dot{Y}_{11} \\ \dot{Y}_{21} \\ \dot{Y}_{31} \\ \dot{Y}_{41} \\ \dot{Y}_{51} \end{bmatrix} = \begin{bmatrix} -0.0297 & & & 0.0438-K_{11} & -K_{12} & Y_{11} \\ 0.331 & -0.0042 & -0.0461 & -K_{21} & -K_{22} & Y_{21} \\ -1.13 & 0.128 & -0.803 & -K_{31} & -K_{32} & Y_{31} \\ 0 & 0 & 1 & -K_{41} & -K_{42} & Y_{41} \\ 0 & 1 & 0 & -K_{51} & -K_{52} & Y_{51} \end{bmatrix}$$

Where we would finally have fifteen simultaneous differential equations as follows:

$$\dot{K}_{11} = 1000 Y_{11} Y_{41}$$

$$\dot{K}_{12} = 1000 Y_{11} Y_{51}$$

$$\dot{K}_{21} = 1000 Y_{21} Y_{41}$$

$$\dot{K}_{22} = 1000 Y_{21} Y_{51}$$

$$\dot{K}_{31} = 1000 Y_{31} Y_{41}$$

$$\dot{K}_{32} = 1000 Y_{31} Y_{51}$$

$$\dot{K}_{41} = 1000 Y_{41} Y_{41}$$

$$\dot{K}_{42} = 1000 Y_{41} Y_{51}$$

$$\dot{K}_{51} = 1000 Y_{51} Y_{41}$$

$$\dot{K}_{52} = 1000 Y_{51} Y_{51}$$

$$\dot{Y}_{11} = 0.0297 Y_{11} - Y_{21} + 0.0438 Y_{41} - K_{11} Y_{41} - K_{12} Y_{51}$$

$$\dot{Y}_{21} = 0.331 Y_{11} - 0.0042 Y_{21} - 0.0461 Y_{31} - K_{21} Y_{41} - K_{22} Y_{51}$$

$$\dot{Y}_{31} = 1.13 Y_{11} + 0.128 Y_{21} - 0.803 Y_{31} - K_{31} Y_{41} - K_{32} Y_{51}$$

$$\dot{Y}_{41} = Y_{31} - K_{41} Y_{41} - K_{42} Y_{51}$$

$$\dot{Y}_{51} = Y_{21} - K_{51} Y_{41} - K_{52} Y_{51}$$

I have used IBM-1130 to calculate the values of Kalman gain.

Fig. 3 shows the plot of these values. We see that the solution of the Chandrasekhar-type equations run quire smoothly into steady state.

The complete computer program and numerical results, using the 4th order Runge-Kutta method, are shown in Appendix I.

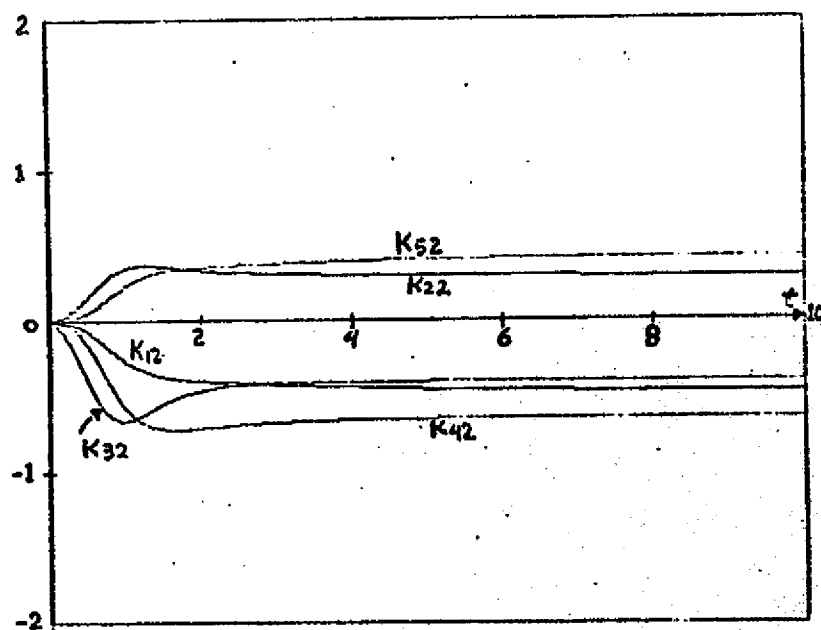
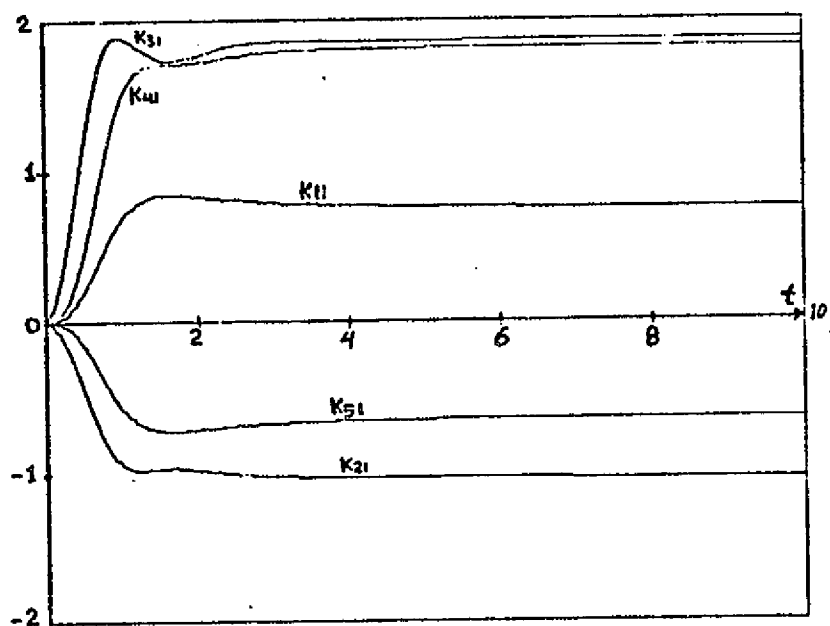


Fig. 3. Plots of Kalman gain matrix computed using Chandrasekhar-type equations for example in Case I.

EXAMPLE: 4

As a further example consider an RLC circuit as shown in Fig. 4. The voltage $Z(t)$ is the out put of a system with input, the white noise $U(t)$. By Kirchhoff's law the following voltage equation may be written for the circuit

$$U(t) = L \frac{di}{dt} + Ri(t) + \frac{1}{C} \int_{t_1}^t i dt \quad (2-30)$$

where R , L , and C are the resistance, inductance, and capacitance, respectively, and $i(t)$ is the circuit current.

Let the variable q be the electric charge then:

$$\frac{dq}{dt} \equiv i(t) \quad (2-31)$$

$$q(t_1) = 0 \quad (2-32)$$

Equation (2-30) may be written

$$U(t) = L \frac{d^2q}{dt^2} + R \frac{dq}{dt} + \frac{1}{C} q \quad (2-33)$$

Equation (2-33) is a second-order equation governing the behavior of charge density.

To put Equation (2-33) in the standard state-variable form use is made of the definition of Equation (2-31). Thus Equation (2-33) is equivalent to the two coupled first-order equations.

$$\frac{dq}{dt} = i(t)$$

$$L \frac{di}{dt} = - Ri(t) - \frac{1}{C} q + U(t) \quad (2-34)$$

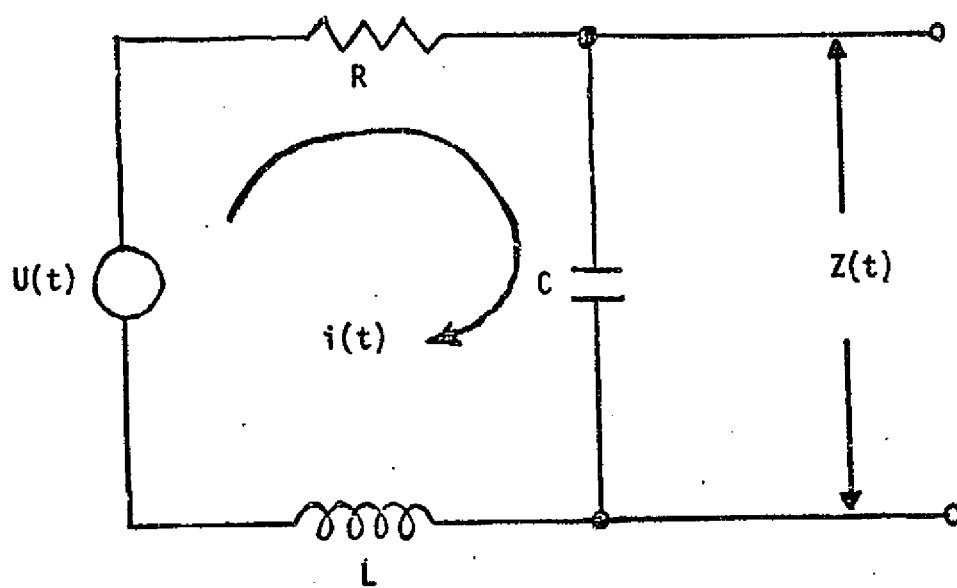


FIG. 4. AN RLC CIRCUIT

Defining

$$X_1 \equiv q \quad X_2 \equiv i(t) \quad (2-35)$$

Equation (2-34) may be written as

$$\dot{X}(t) = F X(t) + GU(t) \quad (2-36)$$

where

$$X(t) = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \quad U(t) = [U_1], \quad F = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}$$

Similarly output voltage $Z(t)$ may be written as

$$Z(t) = \frac{1}{C} q = \frac{1}{C} X_1$$

or

$$Z(t) = H X(t), \quad H = \left[\frac{1}{C} \quad 0 \right], \quad (2-37)$$

Suppose that our measurement is $y(t)$

$$y(t) = Z(t) + v(t), \quad v(t) \equiv \text{observable noise} \quad (2-38)$$

Let us assume that the input noise and the observable noise are uncorrelated, so that

$$E[U(t) v'(s)] = 0 \quad (2-39a)$$

and

$$E[U(t) U'(s)] = [1] \quad \pi_0 = 0 \quad (2-39b)$$

With these assumptions we can write Chandrasekhar-type equations as follow

$$\dot{K}_{11} = \frac{1}{C} Y_{11}^2$$

$$\dot{K}_{21} = \frac{1}{C} Y_{11} Y_{21}$$

$$\dot{Y}_{11} = -\frac{1}{C} K_{11} Y_{11} + Y_{21}$$

$$\dot{Y}_{21} = \left(-\frac{1}{LC} - \frac{1}{C} K_{21} \right) Y_{11} - \frac{R}{L} Y_{21}$$

$$L_1 = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}$$

Fig. 5 shows the plot of the Kalman gain for this problem where we assumed that

$$C = 0.01 \quad \text{Farad}$$

$$R = 10.0 \quad \text{Ohm}$$

$$L = 1.0 \quad \text{Henry}$$

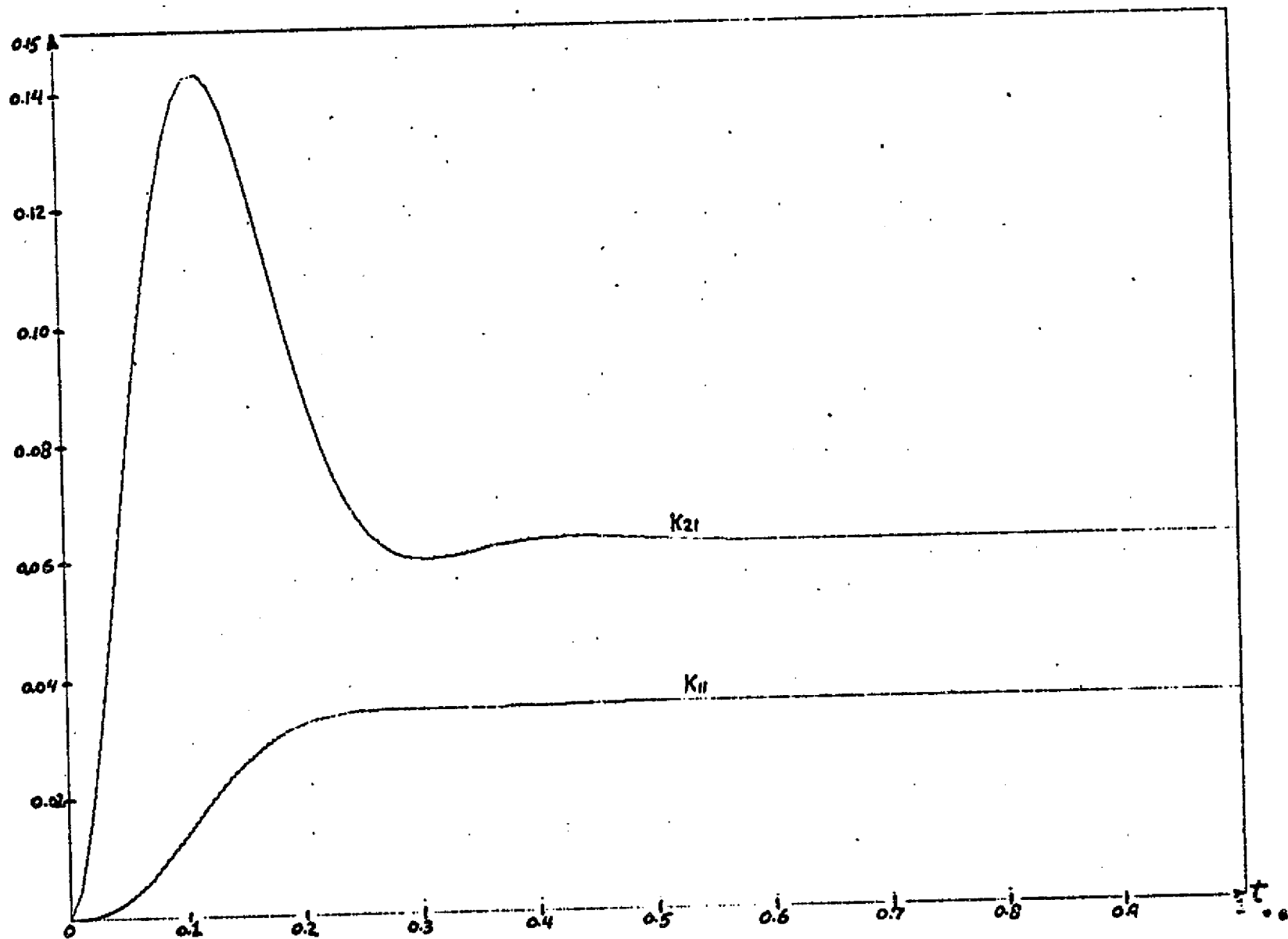


Fig. 5. Plot of Kalman Gain Matrix

Case II: (High Initial Uncertainty)

If the initial state $X(t_0)$ is not known, then we may assume that

$$E[x_0 x_0] = \Pi_0 = \infty \quad (2-40)$$

In this case what we should do for both Riccati-type and Chandrasekhar-type equations is work with $P^{-1}(t)$ instead of $P(t)$.

Using equation (2-10) we can get the relation

$$\frac{d}{dt} [P^{-1}(t)] = -P^{-1}(t)\dot{P}(t)P^{-1}(t) \quad (2-41)$$

Substitute $\dot{P}(t)$ into the above equation, then we have

$$\begin{aligned} \frac{d}{dt} [P^{-1}(t)] = & -P^{-1}(t)F_c - F_c^{-1}P^{-1}(t) + H'H - P^{-1}(t) \\ & GQ_cG'P^{-1}(t) \end{aligned} \quad (2-42)$$

where we assumed that

$$F_c = F - GCH \quad Q_c = Q - CC' \quad (2-43)$$

And if we let

$$S(t) = P^{-1}(t) \quad (2-44)$$

$$\hat{R}(t) = P^{-1}(t)\hat{X}(t) \quad (2-45)$$

Then the Kalman-filter equation can be written as

$$\dot{\hat{R}}(t) = P^{-1}(t)\dot{\hat{X}}(t) + \frac{d}{dt} [P^{-1}(t)]\hat{X}(t) \quad (2-46)$$

or

$$\dot{\hat{R}}(t) = -[F_c + GQ_cG'S(t)]\hat{R}(t) + H'Y(t), \quad \hat{R}(t_0) = 0 \quad (2-47)$$

And from (2-42) we have

$$\begin{aligned}\dot{S}(t) &= -S(t)F_c - F_c^T S(t) - S(t)GQ_c G^T S(t) + H^T H, \\ S(t_0) &= P^{-1}(t_0)\end{aligned}\quad (2-48)$$

The equation (2-44) - (2-48) are called "information-filter" form of the Kalman filter. And whenever we have problems with large initial uncertainty, it is convenient to use them because

$$\Pi_0 = P(t_0) = 0 \rightarrow S(t_0) = 0$$

To obtain Chandrasekhar-type equation, let

$$GQ_c G^T = L_1 L_1^T \quad (2-49)$$

Now let us define an $\alpha \times n$ matrix $A(t)$ such that

$$A(t) = L_1^T S(t) \quad (2-50)$$

Using the above equation, then the Kalman-filter equation (2-47) will become

$$\dot{\hat{R}}(t) = -[F_c + L_1 A(t)]^T \hat{R}(t) + H^T Y(t), \quad \hat{R}(t_0) = 0 \quad (2-51)$$

and it can be shown that $A(t)$ in (2-51) can be obtained via the following equations

$$\dot{A}(t) = L_1^T B(t) B^T(t), \quad A(t_0) = 0 \quad (2-52)$$

$$\dot{B}(t) = -[F_c + L_1 A(t)]^T B(t), \quad B(t_0) = H^T \quad (2-53)$$

where $S(t)$ can be calculated via the equation

$$S(t) = \int_0^t B(\tau) B^T(\tau) d\tau \quad (2-54)$$

Case III: (Stationary Processes)

In this case we assume that

$$E[x_0 x_0'] = \Pi = \bar{\Pi} \quad (2-55)$$

Since F is a stability matrix and the eigenvalues of F have negative real parts, then as $|t - t_0| \rightarrow \infty$, $Z(\cdot)$ will reduce to a stationary process.

It can be shown, as was first done by Doob (1944), that for $Z(\cdot)$ as in (2-3) - (2-6) with constant parameters

$$E[Z(t) Z'(s)] = H E[X(t) X'(s)] H' \quad (2-56)$$

where we have the relation

$$E[X(t) X'(s)] = \begin{cases} e^{F(t-s)} \Pi(s), & t \geq s \\ \Pi(t) e^{F'(s-t)}, & t \leq s \end{cases} \quad (2-57)$$

Consider the relations

$$\frac{d}{dt} \phi(t, t_0) = F \phi(t, t_0) \quad (2-58)$$

$$\frac{d}{dt} \phi'(t, t_0) = \phi'(t, t_0) F' \quad (2-59)$$

$$\frac{d}{dt} \int_0^t f(t, \tau) d\tau = f(t, t) + \int_0^t \frac{d}{dt} f(t, \tau) d\tau \quad (2-60)$$

where $\phi(t, t_0)$ is the transition matrix, then it is not hard to show that the variance matrix $\Pi(t) = E[X(t)X'(t)]$ obeys a linear matrix differential equation

$$\dot{\Pi}(t) = F\Pi(t) + \Pi(t)F' + GQG', \quad \Pi(t_0) = \Pi_0 \quad (2-61)$$

and an explicit solution to the above differential equation is

$$\Pi(t) = e^{F(t-t_0)} \Pi_0 e^{F'(t-t_0)} + \int_{t_0}^t e^{F(t-\tau)} GQG' e^{F'(t-\tau)} d\tau \quad (2-62)$$

Now by previous assumption that F is a stability matrix, we see that as t goes to ∞ , $\Pi(t)$ tends to a constant matrix $\bar{\Pi}$. In fact from (2-61) it is clear that $\bar{\Pi}$ is the unique solution of the equation

$$F\bar{\Pi} + \bar{\Pi}F' + GQG' = 0 \quad (2-63)$$

This is called matrix Lyapunov equation.

To obtain Chandrasekhar equation when $\Pi_0 = \bar{\Pi}$, we proceed as follow:

$$D = -(\bar{\Pi}H' + GC)(\bar{\Pi}H' + GC)' \quad (2-64)$$

and when $\alpha = P$ we can take

$$L_2 = \bar{\Pi}H' + GC \quad (2-65)$$

so that the Chandrasekhar-type equations become

$$\dot{K}(t) = -Y_2(t)Y_2'(t)H', \quad K(t_0) = \bar{\Pi}H' + GC \quad (2-66)$$

$$\dot{Y}_2(t) = [F - K(t)H]Y_2(t), \quad Y_2(t_0) = L_2 \quad (2-67)$$

And also note that $P(t)$ can be found via

$$\dot{P}(t) = -Y_2(t)Y_2'(t), \quad P(t_0) = \bar{\Pi} \quad (2-68)$$

And therefore

$$P(t) = \bar{\Pi} - \int_0^t Y_2(\tau) Y_2'(\tau) d\tau \quad (2-69)$$

Case IV: (Stationary Processes with known Covariances)

This case is actually very close to the usual assumptions of statistical communication theory. Let us assume that the stationary signal and noise processes are given not via a state-space model but by their covariance functions. So that we assume

$$y(t) = Z(t) + v(t), \quad t \geq t_0 \quad (2-70)$$

where we have

$$E[v(t)v'(s)] = I\delta(t - s) \quad (2-71)$$

$$E[Z(t)v'(s)] = 0 \quad s > t \quad (2-72)$$

$$E[y(t)y'(s)] = I\delta(t - s) + K(t, s) \quad (2-73)$$

where $K(t, s)$ is given by

$$K(t, s) = E[Z(t)Z'(s) + Z(t)v'(s) + v(t)Z'(s)] \quad (2-74)$$

$$= \begin{cases} M e^{F(t-s)} N, & t \geq s \\ N' e^{F'(s-t)} M', & t \leq s \end{cases} \quad (2-75)$$

where:

F is an $(n \times n)$ matrix

M is an $(p \times n)$ matrix

N is an $(n \times p)$ matrix

To make this consistent with Case III we shall assume that

$$N \text{ has rank } \alpha, \alpha \leq \min(n, p) \quad (2-76)$$

Note that if we assume that $Z(\cdot)$ and $v(\cdot)$ are uncorrelated, $K(t, s)$ is a covariance function, but here we assume a (one-sided) dependence of signal on past signal and noise, therefore $K(t, s)$ itself will not, in general, be a covariance we remark that such a one-sided dependence arises naturally in feedback control and communication problems.

Generally it is not easy to find a solution for this problem without first having to determine models for $Z(\cdot)$ and $y(\cdot)$. Therefore let us assume the following equations:

$$\dot{\hat{Z}}(t) = M\hat{\theta}(t) \quad (2-77)$$

$$\dot{\hat{\theta}}(t) = F\hat{\theta}(t) + K(t) [Y(t) - M\hat{\theta}(t)] \quad (2-78)$$

$$K(t) = N - \Sigma(t)M' \quad (2-79)$$

where the $n \times n$ matrix $\Sigma(\cdot)$ obeys the Riccati equation

$$\dot{\Sigma}(t) = F\Sigma(t) + \Sigma(t)F' + K(t)K'(t), \quad \Sigma(t_0) = 0 \quad (2-80)$$

Under the assumption that the matrices M , F , and N are constant, we may calculate $K(\cdot)$ via the following $n(p + p_1)$ Chandrasekhar-type equations:

$$\dot{K}(t) = -Y(t)Y'(t)M' \quad (2-81)$$

$$\dot{Y}(t) = (F - K(t)M) Y(t) \quad (2-82)$$

$$K(t_0) = N \quad Y(t_0) = N_0 \quad (2-83)$$

where the initial conditions N and N_0 satisfy

$$NN' = N_0N_0' \quad (2-84)$$

and

$$N_0 = n \times p_1 \text{ matrix} \quad (2-85)$$

Note that whenever $\alpha = p$, where α is rank of NN' , we have much simpler formula, since we can take

$$Y(t_0) = K(t_0) = N \quad (2-86)$$

Also note that there Chandrasekhar equations have the same form as in Case I.

CHAPTER III

CONNECTION BETWEEN THE CHANDRASEKHAR X, Y
FUNCTIONS AND THE CHANDRASEKHAR-TYPE EQUATIONS

The algorithm is now in a form where we can point out a close relationship to some famous equations obtained in astrophysics. It is shown that the Kalman gain function was intimately related to the basic X and Y functions of Chandrasekhar (1).

Consider the stochastic process

$$y(t) = Z(t) + v(t), \quad 0 \leq t \leq T \leq \infty \quad (3-1)$$

where v is a white Gaussian noise process with unit spectral intensity, such that

$$E[v(t)] = 0, \quad E[v(t)v'(s)] = \delta(t - s) \quad (3-2)$$

and $Z(t)$ is such that

$$E[Z(t)v'(s)] = 0, \quad t < s \quad (3-3)$$

And v and Z are n -dimensional vectors. Also for simplicity we assume that

$$E[Z(t)] = 0, \quad 0 \leq t \leq T$$

Therefore the covariance K of the signal process Z is the $n \times n$ matrix

$$K(t, s) = E[Z(t)Z'(s)], \quad 0 \leq t, s \leq T, \quad (3-4)$$

where we impose the condition that $K(t, s)$ is continuous in t and s .

Let us define the quantity $\hat{Z}(t)$ to be the linear least squares estimate of $Z(t)$ to be given the observation $\{y(s), 0 \leq s \leq t\}$, $0 \leq t \leq T$.

It is now a well-known fact that $\hat{Z}(t)$ may be written as a linear functional of Y ,

$$\hat{Z}(t) = \int_0^t h(t,s)y(s)ds, \quad (3-5)$$

where the weighting function $h(t,s)$ satisfies the integral equation

$$h(t,s) = K(t,s) - \int_0^t K(\tau,s)h(t,\tau)d\tau, \quad 0 \leq s \leq t \leq T \quad (3-6)$$

In view of (3-5), all the information necessary to characterize the optimal estimate is contained in the function $h(t,s)$. Let us now turn our attention toward development of a procedure for obtaining h as a function of the observation time t .

Consider the integral equation

$$h(t,s) = K(t,s) - \int_0^t K(\tau,s)h(t,\tau)d\tau, \quad 0 \leq s \leq t \leq T \quad (3-7)$$

where the kernel K is subject to the previous assumptions. We now wish to impose the additional assumption that K may be represented in the form:

$$K(t,s) = K(|t-s|) = \int_0^1 \exp(-|t-s|\alpha)\omega(\alpha)d\alpha \quad (3-8)$$

and, by suitable choice of ω (sums of delta functions, for example), a broad class of important problems may be simply handled. To avoid trivial details, we shall assume ω is a scalar function.

We can show that an initial value representation for h is given in terms of the functions X , Y and J as

$$\frac{\partial X(t, \alpha)}{\partial t} = -Y(t, \alpha) \int_0^1 Y(t, \alpha') \omega(\alpha') d\alpha' \quad (3-9)$$

$$\frac{\partial Y(t, \alpha)}{\partial t} = -\alpha Y(t, \alpha) - X(t, \alpha) \int_0^1 Y(t, \alpha') \omega(\alpha') d\alpha' \quad (3-10)$$

$$\frac{\partial J(s, t, \alpha)}{\partial t} = -\alpha J(s, t, \alpha) - X(t, \alpha) \int_0^1 J(s, t, \alpha') \omega(\alpha') d\alpha' \quad (3-11)$$

for $0 \leq s \leq t \leq T$, $0 \leq \alpha \leq 1$.

$$X(0, \alpha) = 1 \quad (3-12)$$

$$Y(0, \alpha) = 1 \quad (3-13)$$

$$J(s, s, \alpha) = X(s, \alpha) \quad (3-14)$$

The function h is then given by

$$h(t, s) = \int_0^1 J(s, t, \alpha') \omega(\alpha') d\alpha' \quad (3-15)$$

The two nonlinear differential equations (3-9) and (3-10) are generally known as Chandrasekhar's X and Y functions.

Although knowledge of the function $h(t, s)$ is sufficient to determine the optimal estimate $\hat{Z}(t)$ via (3-5), it is computationally desirable to obtain an alternative form for \hat{Z} involving a quadrature over a fixed interval, rather than the variable interval of (3-5). Such a representation will also simplify the real-time calculation of \hat{Z} .

Recall (3-5):

$$\hat{Z}(t) = \int_0^t h(t, s) Y(s) ds, \quad t \geq 0$$

Making use of (3-15), we see that

$$\hat{Z}(t) = \int_0^t \int_0^1 J(s, t, \alpha') \omega(\alpha') d\alpha' y(s) ds \quad (3-16)$$

Interchanging the order of integration in (3-16) gives

$$\hat{Z}(t) = \int_0^1 \int_0^t J(s, t, \alpha') y(s) ds \omega(\alpha') d\alpha' \quad (3-17)$$

Introduce the new function $L(t, \alpha)$ as

$$L(t, \alpha) = \int_0^t J(s, t, \alpha) y(s) ds, \quad t \geq 0, \quad 0 \leq \alpha \leq 1 \quad (3-18)$$

We can show that L actually satisfies the Cauchy problem:

$$\frac{\partial}{\partial t} L(t, \alpha) = -\alpha L(t, \alpha) + X(t, \alpha) \left\{ y(t) - \int_0^1 L(t, \alpha') \omega(\alpha') d\alpha' \right\} \quad (3-19)$$

$$L(t, 0) = 0 \quad (3-20)$$

By using these results, the optimal estimate \hat{Z} is then given by

$$\hat{Z}(t) = \int_0^1 L(t, \alpha') \omega(\alpha') d\alpha' \quad (3-21)$$

At this point it seems essential to point out that unless the function $h(t, s)$ is desired for some reason other than calculating \hat{Z} , we need not calculate it at all. In fact, (3-21) and (3-19) show that knowledge of the function X and L suffice to determine \hat{Z} . This observation is of considerable computational importance, as well as of some analytical interest.

With some effort, we should be able to see that these equations are essentially the same as (2-77), (2-78), (2-81), (2-82), (2-86) if we make the assumption

$$W(\alpha) = \sum_1^n \alpha_i \delta(\alpha - \alpha_i), \alpha_i \geq 0, -F = \text{diag}(\alpha_1, \dots, \alpha_n)$$

This is why the equation for $K(\cdot)$ and $Y(\cdot)$ are said to be of Chandrasekhar type.

Further Examine of Case IV

To make some connections with the radiative-transfer problem in which Chandrasekhar originally introduced the $X(\cdot)$ and $Y(\cdot)$ functions, we shall examine Special Case IIIa further. Let us assume also that N has rank P , so that the relevant equations are (2-81), (2-82), and (2-86). Now let us define an impulse response function $h(t, \tau)$ as

$$\hat{Z}(t) = \int_0^t h(t, \tau) Y(\tau) d\tau \quad (3-22)$$

Then we can show that

$$h(t, \tau) = \begin{cases} M\psi(t, \tau)K(\tau), & t \geq \tau \\ 0, & t \leq \tau \end{cases} \quad (3-23)$$

where $\psi(\cdot, \cdot)$ is the state-transition matrix of $F - K(\cdot)M$, that is

$$\frac{d\psi(t, \tau)}{dt} = (F - K(t)M) \psi(t, \tau), \psi(\tau, \tau) = I \quad (3-24)$$

Also note that $h(\cdot, \cdot)$ satisfies the Wiener-Hopf type of equation

$$h(t, \tau) + \int_0^t h(t, s) K(s, \tau) ds = K(t, \tau), t \geq \tau \quad (3-25)$$

Now it is clear that from (3-23) and (2-7), (2-82) we can obtain the following relations

$$h(t, t) = MK(t) \quad (3-26)$$

$$h(t, 0) = M\psi(t, 0)K(0) = MY(t) \quad (3-27)$$

Then we see that the equation (2-81)

$$\dot{K}(t) = -Y(t)Y'(t)M', \text{ is the same as}$$

$$\frac{\partial h(t, t)}{\partial t} = -h(t, 0) h'(t, 0) \quad (3-28)$$

which is a result that follows from a resolvent identifying in the theory of integral equations.

It is of interest to note that when the signal $Z(\cdot)$ and noise $v(\cdot)$ are uncorrelated, i.e., when $K(t, s)$ is a covariance function, we can identify

$$MK(t) = E[\tilde{Z}(t) \tilde{Z}'(t)] \quad (3-29)$$

$$MY(t) = E[\tilde{Z}(t) \tilde{Z}'(0)] \quad (3-30)$$

where $\tilde{Z}(t) = \hat{Z}(t) - Z(t)$ is the instantaneous error in the estimate of $Z(t)$. The relations (3-29) and (3-30) have been used to suggest filtering-theory analogs of some quantities that arise in radiative transfer, e.g., the reflection, transmission, and internal intensity functions; however, these analogies have to be treated with some care because the assumption that $K(t, s)$ is a covariance function, which is

necessary for (3-29) and (3-30), is not valid in the radiative-transfer problem.

CHAPTER IV

PROOF OF THE CHANDRASEKHAR-TYPE FORMULAS

The Chandrasekhar-type equations presented in previous chapters can be derived in several ways. We choose one that begins with knowledge of the Kalman filter solution. In this way, the manner in which the constancy of the model parameters can be exploited will perhaps stand out most clearly. Also, all questions of existence and uniqueness can be resolved immediately by use of the known results for the Kalman Theory (5).

For our convenience, let us restate the equations of the Kalman filter

$$\dot{\hat{X}}(t) = F\hat{X}(t) + K(t)[Y(t) - H\hat{X}(t)], \quad \hat{X}(t_0) = 0 \quad (4-1)$$

$$K(t) = P(t)H' + GQ \quad (4-2)$$

$$\begin{aligned} \dot{P}(t) &= FP(t) + P(t)F' - K(t)K'(t) + GQG', \\ P(t_0) &= \Pi_0 \end{aligned} \quad (4-3)$$

Let us define $\psi(t, t_0)$ to be the state-transition matrix of the matrix $F - K(t)H$. That is $\psi(t, t_0)$ is the unique solution of the linear

$$\frac{d\psi(t, t_0)}{dt} = [F - K(t)H] \psi(t, t_0), \quad \psi(t_0, t_0) = I \quad (4-4)$$

Before we proceed to derive our algorithm, let us consider the following lemma

Lemma: The derivative $\dot{P}(\cdot)$ of the solution $P(\cdot)$ of the Riccati-type equation (2-12) can be written as follow

$$\dot{P}(t) = \psi(t, t_0) \dot{P}(t_0) \psi'(t, t_0) \quad (4-5)$$

where

$$\dot{P}(t_0) = D = FP(t_0) + P(t_0)F' - K(t_0)K'(t_0) + GQG' \quad (4-6)$$

And if we let $P(t_0) = \Pi_0$ and $K(t_0) = \Pi_0 H' + GC$ we have:

$$\dot{P}(t_0) = D = F\Pi_0 + \Pi_0 F' - (\Pi_0 H' + GC)(\Pi_0 H' + GC)' + GQG' \quad (4-7)$$

Proof: we attack (4-3) by differentiation to get

$$\ddot{P}(t) = F\dot{P}(t) + \dot{P}(t)F' - K(t)\dot{K}'(t) - \dot{K}(t)K'(t) \quad (4-8)$$

Also, differentiation of both sides of (4-2) yield:

$$\dot{K}(t) = \dot{P}(t) H' \quad (4-9)$$

Using equation (4-9), we can rearrange (4-8) as

$$\ddot{P}(t) = (F - K(t) H) \dot{P}(t) + \dot{P}(t)(F - K(t) H)' \quad (4-10)$$

Now let us temporarily disregard the dependence of $K(\cdot)$ on $P(\cdot)$, and just regard $K(\cdot)$ as some given function, then equation (4-10) is a linear homogeneous equation in $\dot{P}(\cdot)$. This equation can be solved to give

$$\dot{P}(t) = \psi(t, t_0) \dot{P}(t_0) \psi'(t, t_0)$$

It can be easily checked by differentiation that the last expression for $P(\cdot)$ does indeed satisfy (4-10).

The basic idea underlying the rest of the arguments is that for certain choices of $P(t_0)$, $\dot{P}(t_0)$, and therefore $\dot{P}(t)$ can have rank less than n . For example let us consider Case I.

Proof of the Case I

Since we assumed that $\Pi_0 = 0$, therefore (4-7) becomes

$$\dot{P}(t_0) = GQ_C G',$$

where

$$Q_C = Q - CC'$$

And the basic relation (4-5) becomes

$$\dot{P}(t) = \psi(t, t_0) GQ_C G' \psi'(t, t_0) \quad (4-11)$$

Since $GQ_C G'$ is nonnegative-definite, we can factor it as

$$GQ_C G' = L_1 L_1', \quad L_1 = n \times \alpha \text{ matrix} \quad (4-12)$$

where

$$\text{rank of } GQ_C G' = \alpha$$

Note that the dimension of L_1 is unique. Now if we define

$$Y_1(t) = \psi(t, t_0) L_1 \quad (4-13)$$

Then by differentiation we see that

$$\dot{Y}_1(t) = [F - K(t)H] Y_1(t), \quad Y_1(t_0) = L_1 \quad (4-14)$$

Also from (4-11) - (4-13) we see that

$$\dot{P}(t) = Y_1(t) Y_1'(t)$$

Then (4-18) can be written as

$$\dot{P}(t) = Y_1(t) Y_1'(t) - Y_2(t) Y_2'(t) \quad (4-20)$$

And also from (4-19)

$$\dot{Y}_i(t) = (F - K(t) H) Y_i(t), \quad Y_i(t_0) = L_i \quad (4-21)$$

Then we see that

$$\dot{K}(t) = \dot{P}(t)H' = [Y_1(t)Y_1'(t) - Y_2(t)Y_2'(t)] H' \quad (4-22)$$

$$K(t_0) = P(t_0)H' + GC = \Pi_0 H' + GC \quad (4-23)$$

Equations (4-21) - (4-23) are just the Chandrasekhar-type equations, and the proof of the general formula is completed.

Proof of the Case IV

In this case we have covariance information rather than a state model. Let us work with the Riccati equation (2-80).

$$\dot{\Sigma}(t) = F\Sigma(t) + \Sigma(t)F' + K(t)K'(t), \quad \Sigma(t_0) = 0$$

$$K(t) = N - \Sigma(t)M'$$

and thus for $\ddot{\Sigma}(t)$ we can have

$$\ddot{\Sigma}(t) = (F - K(t)M) \dot{\Sigma}(t) + \dot{\Sigma}(t) (F - K(t)M)' \quad (4-24)$$

Now, if we define $\psi(t, t_0)$ to be the state-transition matrix of $F - K(t)M$, then from (4-24) we have

$$\dot{\Sigma}(t) = \psi(t, t_0) \dot{\Sigma}(t_0) \psi'(t, t_0) \quad (4-25)$$

and from (4-9) we have

$$\dot{K}(t) = \dot{P}(t)H' = Y_1(t) Y_1'(t)H' \quad (4-15)$$

$$K(t_0) = P(t_0)H' + GC = GC \quad (4-16)$$

Equations (4-14) - (4-16) are simultaneous Chandrasekhar-type equations for Case I.

Proof of the General Formulas

The proof in the general case is almost as simple. Since for any value of π_0 , $D = \dot{P}(t_0)$ as given in equation (4-7) is symmetric, therefore, it can be factored as

$$D = \dot{P}(t_0) = L_1 L_1' - L_2 L_2' \quad (4-17)$$

Let us assume that

$$\alpha = \text{rank of } D$$

and rank of $L_1 L_1'$ is the number, say β , where β is the number of positive eigenvalues of D and the rank of $L_2 L_2'$ is $\alpha - \beta$.

Here we can always choose the dimensions of L_1 and L_2 so that

$$L_1 = n \times \beta, \quad L_2 = n \times (\alpha - \beta).$$

By these assumptions, now we can write (4-5) as

$$P(t) = \psi(t, t_0) L_1 L_1' \psi'(t, t_0) - \psi(t, t_0) L_2 L_2' \psi'(t, t_0) \quad (4-18)$$

Now if we define

$$Y_i(t) = \psi(t, t_0) L_i, \quad i = 1, 2 \quad (4-19)$$

and from (2-80) we get

$$\dot{\Sigma}(t) = K(t_0) K'(t_0) = \Pi \Pi' \quad (4-26)$$

Now if N has rank P , $\leq \min(n, p)$, we can write

$$N N' = N_0 N_0', \quad N_0 \text{ an } n \times p, \text{ matrix}$$

Therefore from (4-25)

$$\dot{\Sigma}(t) = \psi(t, t_0) N_0 N_0' \psi'(t, t_0) = Y(t) Y'(t) \quad (4-27)$$

where we can take

$$Y(t) = \psi(t, t_0) N_0$$

and therefore

$$\dot{Y}(t) = (F - K(t) M) Y(t), \quad Y(t_0) = N_0 \quad (4-28)$$

Note that when $P_1 = P$ we can identify N and N_0 . Equations (4-26) - (4-28) are Chandrasekhar-type equations.

Also note that we can verify the relation

$$\Sigma(t) = \Pi(t) - P(t)$$

so that

$$\dot{\Sigma}(t) = \dot{\Pi} - \dot{P}(t) = -\dot{P}(t)$$

And if $K(t, s)$ as in (2-75) arises from a model of the form (2-1) - (2-6), we can identify

$$M = H, \quad N(t) = \Pi(t) H' + GC$$

CONCLUSION

A new class of algorithm for linear least squares estimation problem is presented. The new algorithm does have some computational saving when the number of observation vectors are much less than the dimension of the state space.

APPENDIX I

Fortran IV Program for Solution of the Chandrasekhar-Type Equations

Here we present the computer program, using the fourth-order Runge-Kutta method, to solve the Chandrasekhar-type equations of the example 3 in Chapter II, Case I.

The purpose of the Runge-Kutta method is to obtain an approximate solution of a system of first order ordinary differential equations.

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// JCH COOF CCGA COGA

LOG DRIVE	CART SPEC	CART AVAIL	PHY DRIVE
0000	0001	000F	0001
0001	000A	000A	0002
		000F	0000

V2 M11 ACTUAL 16K CONFIG 16K

// CLP

*DELETE CJO1
CART ID CCOF CH ADDR 5946 DH CNT 0021

// FORTRAN

C-ERRS...STNO-C..... F O R T R A N S C R I P T C E S T A T E M E N T S IDENTFCN **COMPILER MESSAGES**

```
C      MAFFCOO OJALALI
      SUBROUTINE CJO1(T,U,CU)
      DIMENSION U(5,3),CU(5,3)
      CL(1,1)=1000.0*U(1,3)*U(4,3)
      CL(1,2)=1000.0*U(1,3)*U(5,3)
      CL(2,1)=1000.0*U(2,3)*U(4,3)
      CL(2,2)=1000.0*U(2,3)*U(5,3)
      CL(3,1)=1000.0*U(3,3)*U(4,3)
      CL(3,2)=1000.0*U(3,3)*U(5,3)
      CL(4,1)=1000.0*U(4,3)*U(4,3)
      CL(4,2)=1000.0*U(4,3)*U(5,3)
      CL(5,1)=1000.0*U(5,3)*U(4,3)
      CL(5,2)=1000.0*U(5,3)*U(5,3)
      CL(1,2)=-0.0297*U(1,3)-U(2,3)+0.0430*U(4,3)
1      -U(1,1)*U(4,3)-U(1,2)*U(5,3)
      CL(2,3)=0.331*U(1,3)-0.0042*U(2,3)-0.0461*U(3,3)
2      -U(2,1)*U(4,3)-U(2,2)*U(5,3)
      CL(3,3)=-1.13*U(1,3)+0.120*U(2,3)-0.803*U(3,3)
3      -U(3,1)*U(4,3)-U(3,2)*U(5,3)
      CL(4,3)=U(3,3)-U(4,1)*U(4,3)-U(4,2)*U(5,3)
      CL(5,3)=U(2,3)-U(5,1)*U(4,3)-U(5,2)*U(5,3)
      RETURN
      END
```

FEATURES SUPPORTED
ONE WORD INTEGERS
STANDARD PRECISION
SAVE

CORE REQUIREMENTS FOR - CJO1
COMMON- U, VARIABLES AND TEMPORARIES- 24, CONSTANTS AND PROGRAM- 440

RELATIVE ENTRY POINT ADDRESS IS 002A (HEX)

END OF SUCCESSFUL COMPILATION

*STORE WS LA CJO1 ** THIS COMPILATION SAVED SUCCESSFULLY UNLESS A 'C' ERROR FOLLOWS **
CART ID CCOF CH ADDR 5979 DH CNT 0021

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OF POOR QUALITY

// JCB 000F CCGA C00A

LOG DRIVE	CART SPEC	CART AVAIL	PHY DRIVE
0000	000F	000F	0001
0001	000A	000A	0007
		000H	0000

V2 M11 ACTUAL 18K CONFIG 16K

// PUP

*DELETE CJOZ
 CART ID CCOF CR ADDR 5946 DB CNT C00C

// FORTRAN

C-ERKS...SIND.C..... F O R T R A N S O U R C E S T A T E M E N T S IDENTFCN **COMPILER MESSAGES**

SUBROUTINE CJOZ(T,U,CU,IHLF,NL,PRMT)
 DIMENSION U(5,3),CU(5,3),PRMT(5)
 DATA NXREG/1/, IHOLD/0/
 DATA NREC/0/

C
C
C

IDEL = IHLF - IHOLD
 IF (IDEL .EQ. 0) GO TO 3
 IF (IDEL .LT. 0) GO TO 1
 NXREG = NXREG + 2 * IDEL
 GO TO 2
1 CONTINUE
 NXREG = NXREG / 2
2 CONTINUE
 IHOLD = IHLF
3 CONTINUE
 NXREG = NXREG - 1
 IF (NXREG .GT. 0) GO TO 4
 TP = T + 0.0005
 WRITE(5,100) TP, U, IHLF
 NREC = NREC + 1
 WRITE(1,NREC) T, ((U(I,J), I=1,5), J=1,2)
 NXREG = 2 * IHLF
4 CONTINUE
 RETURN

C
C

100 FORMAT(1X, F6.3, 2X, 10F6.2, 2X, 5F7.3, 15)
 END

FEATURES SUPPORTED
 ONE WORD INTEGERS
 STANDARD PRECISION
 SAVE

CORE REQUIREMENTS FOR - LHM2
 COMMON- J. VARIABLES AND TEMPORARIES- 10. CONSTANTS AND PROGRAM- 164

RELATIVE ENTRY POINT ADDRESS IS 001A (HEX)

END OF SUCCESSFUL COMPILATION

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*STERE WS LA 1J02

CART 10 0001 CP AFER 54BE DB CAT 000C

** THIS COMPILATION SAVED SUCCESSFULLY UNLESS A 'C' ERROR FOLLOWS **

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PAGE 1 10/18/74

// JOB 0001 CCGA 000H CCGH CCGA

LOG DRIVE	CART SPEC	CART AVAIL	PHY DRIVE
0000	000F	000F	0001
0001	000A	000A	0002
0002	000F	000F	0000

V2 FILE ACTUAL 100 CONFIG 100

// FORTRAN

*IOCSIDISK)

C-ERRS...STND.C..... FORTRAN SOURCE STATEMENTS IDENTFCN **COMPILER MESSAGES**

```
EXTERNAL DJ01,DJ02
DIMENSION PRM1(5),U(5,3),CU(5,3),WORK(8,15),UMAG(5,3)
DEFINE FILE 1(112,22,U,ICOPY)
DATA UMAG/2.0,4.0,3.0,3.0,2.0,4.0,2.0,2.0,2.0,4.0,0.1,0.1,0.1,0.1,
4      C.1/
DATA U/0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,
1      0.001,-0.005,0.124,0.0,0.0/
DATA NU/15/
DATA TOL/0.001/
DATA PRM1/0.0,10.0,0.1,2*0.0/
```

```
WRITE(5,2)
CALL RKSCU(NU,UMAG,CL,TOL,PRM1)
CALL RKGS(PRM1,U,CU,NU,INLF,DJ01,DJ02,WORK)
CALL EXIT
```

```
2      FORMAT(1H1,3X,'T',6X,'K11',3X,'K21',3X,'K31',3X,'K41',3X,'K51',3X,
6      'K12',3X,'K22',3X,'K32',3X,'K42',3X,'K52',5X,'Y11',4X,'Y21',4X,'Y3
71',4X,'Y41',4X,'Y51'//)
END
```

FEATURES SUPPORTED
ONE WORD INTEGERS
STANDARD PRECISION
SAVE
IOCS-
DISK
1403 PRINTER
CARD

CORE REQUIREMENTS FOR -
COMMON- 0, VARIABLES AND TEMPORARIES- 354, CONSTANTS AND PROGRAM- 118

END OF SUCCESSFUL COMPILATION

// XEQ 1

*FILES(1,DJ03)

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OF POOR QUALITY

0.000	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.003	-0.035	0.124	0.000	0.000	0
0.100	0.00	-0.02	0.07	0.00	-0.00	-0.00	0.00	-0.02	-0.00	0.006	-0.035	0.113	0.011	-0.003	0
0.200	0.01	-0.08	0.25	0.03	-0.01	-0.00	0.02	-0.07	-0.01	0.010	-0.035	0.103	0.022	-0.007	0
0.300	0.05	-0.15	0.52	0.11	-0.03	-0.01	0.05	-0.16	-0.03	0.013	-0.035	0.092	0.032	-0.010	0
0.400	0.10	-0.30	0.84	0.26	-0.07	-0.03	0.07	-0.26	-0.07	0.016	-0.034	0.080	0.040	-0.013	0
0.500	0.16	-0.44	1.15	0.43	-0.14	-0.06	0.14	-0.37	-0.14	0.019	-0.031	0.067	0.045	-0.016	0
0.600	0.22	-0.59	1.46	0.65	-0.22	-0.09	0.20	-0.48	-0.22	0.021	-0.028	0.057	0.048	-0.018	0
0.700	0.30	-0.72	1.66	0.90	-0.32	-0.14	0.25	-0.57	-0.32	0.022	-0.024	0.037	0.049	-0.019	0
0.800	0.39	-0.83	1.80	1.13	-0.41	-0.18	0.29	-0.63	-0.41	0.022	-0.019	0.022	0.046	-0.019	0
0.900	0.50	-0.90	1.87	1.32	-0.50	-0.22	0.32	-0.66	-0.50	0.021	-0.014	0.009	0.041	-0.019	0
1.000	0.67	-0.95	1.97	1.47	-0.57	-0.26	0.35	-0.66	-0.57	0.019	-0.009	-0.002	0.035	-0.017	0
1.100	0.73	-0.97	1.97	1.58	-0.63	-0.29	0.36	-0.65	-0.63	0.017	-0.005	-0.010	0.028	-0.016	0
1.200	0.78	-0.98	1.92	1.64	-0.67	-0.33	0.36	-0.63	-0.67	0.015	-0.001	-0.017	0.022	-0.014	0
1.300	0.81	-0.98	1.79	1.68	-0.69	-0.34	0.36	-0.60	-0.69	0.013	0.001	-0.021	0.016	-0.012	0
1.400	0.82	-0.98	1.76	1.70	-0.71	-0.36	0.36	-0.58	-0.71	0.011	0.003	-0.024	0.010	-0.011	0
1.500	0.83	-0.97	1.74	1.71	-0.72	-0.37	0.36	-0.55	-0.72	0.010	0.005	-0.025	0.006	-0.009	0
1.600	0.84	-0.97	1.73	1.71	-0.72	-0.38	0.35	-0.53	-0.72	0.009	0.006	-0.025	0.002	-0.008	0
1.700	0.84	-0.97	1.73	1.71	-0.72	-0.38	0.35	-0.50	-0.72	0.008	0.007	-0.025	-0.001	-0.007	0
1.800	0.83	-0.97	1.73	1.71	-0.72	-0.39	0.34	-0.49	-0.72	0.007	0.007	-0.023	-0.003	-0.006	0
1.900	0.83	-0.98	1.74	1.71	-0.72	-0.39	0.33	-0.47	-0.72	0.006	0.007	-0.022	-0.005	-0.006	0
2.000	0.83	-0.98	1.76	1.72	-0.71	-0.40	0.33	-0.46	-0.71	0.005	0.007	-0.020	-0.007	-0.005	0
2.100	0.82	-0.99	1.77	1.72	-0.71	-0.40	0.33	-0.45	-0.71	0.005	0.007	-0.018	-0.008	-0.005	0
2.200	0.82	-0.99	1.79	1.73	-0.71	-0.40	0.32	-0.44	-0.71	0.005	0.007	-0.016	-0.008	-0.004	0
2.300	0.81	-1.00	1.80	1.74	-0.70	-0.40	0.32	-0.43	-0.70	0.005	0.006	-0.013	-0.008	-0.004	0
2.400	0.81	-1.00	1.81	1.74	-0.70	-0.41	0.32	-0.43	-0.70	0.005	0.005	-0.011	-0.009	-0.004	0
2.500	0.80	-1.01	1.82	1.75	-0.69	-0.41	0.31	-0.42	-0.69	0.004	0.005	-0.010	-0.008	-0.004	0
2.600	0.80	-1.01	1.83	1.76	-0.69	-0.41	0.31	-0.42	-0.69	0.004	0.004	-0.008	-0.008	-0.004	0
2.700	0.80	-1.02	1.84	1.77	-0.69	-0.41	0.31	-0.42	-0.69	0.004	0.004	-0.006	-0.008	-0.004	0
2.800	0.79	-1.02	1.84	1.77	-0.68	-0.41	0.31	-0.41	-0.68	0.004	0.003	-0.005	-0.007	-0.004	0
2.900	0.79	-1.02	1.85	1.78	-0.68	-0.42	0.31	-0.41	-0.68	0.004	0.003	-0.004	-0.007	-0.004	0
3.000	0.79	-1.03	1.85	1.78	-0.68	-0.42	0.31	-0.41	-0.68	0.004	0.003	-0.003	-0.006	-0.004	0
3.100	0.78	-1.03	1.85	1.79	-0.67	-0.42	0.30	-0.41	-0.67	0.004	0.002	-0.002	-0.006	-0.004	0
3.200	0.78	-1.03	1.85	1.79	-0.67	-0.42	0.30	-0.41	-0.67	0.004	0.002	-0.002	-0.005	-0.004	0
3.300	0.78	-1.03	1.85	1.79	-0.67	-0.42	0.30	-0.41	-0.67	0.004	0.002	-0.001	-0.005	-0.004	0
3.400	0.77	-1.03	1.85	1.80	-0.67	-0.43	0.30	-0.41	-0.67	0.004	0.001	-0.001	-0.004	-0.004	0
3.500	0.77	-1.03	1.85	1.80	-0.67	-0.43	0.30	-0.41	-0.67	0.004	0.001	-0.001	-0.004	-0.004	0
3.600	0.77	-1.03	1.86	1.80	-0.66	-0.43	0.30	-0.41	-0.66	0.004	0.001	-0.001	-0.003	-0.004	0
3.700	0.77	-1.03	1.86	1.80	-0.66	-0.43	0.30	-0.40	-0.66	0.004	0.001	-0.000	-0.003	-0.003	0
3.800	0.77	-1.03	1.87	1.80	-0.66	-0.43	0.30	-0.40	-0.66	0.004	0.001	-0.000	-0.003	-0.003	0
3.900	0.77	-1.03	1.86	1.80	-0.66	-0.44	0.30	-0.40	-0.66	0.004	0.001	-0.000	-0.003	-0.003	0
4.000	0.76	-1.03	1.86	1.80	-0.66	-0.44	0.30	-0.40	-0.66	0.004	0.001	-0.000	-0.002	-0.003	0
4.100	0.76	-1.03	1.86	1.81	-0.66	-0.44	0.30	-0.40	-0.66	0.004	0.001	-0.000	-0.002	-0.003	0
4.200	0.76	-1.03	1.86	1.81	-0.66	-0.44	0.30	-0.40	-0.66	0.004	0.001	-0.000	-0.002	-0.003	0
4.300	0.76	-1.04	1.86	1.81	-0.66	-0.44	0.30	-0.40	-0.66	0.004	0.001	-0.001	-0.002	-0.003	0
4.400	0.76	-1.04	1.86	1.81	-0.65	-0.44	0.30	-0.40	-0.65	0.003	0.001	-0.001	-0.002	-0.003	0
4.500	0.76	-1.04	1.86	1.81	-0.65	-0.45	0.30	-0.40	-0.65	0.003	0.001	-0.001	-0.002	-0.003	0
4.600	0.76	-1.04	1.86	1.81	-0.65	-0.45	0.30	-0.40	-0.65	0.003	0.001	-0.001	-0.002	-0.003	0
4.700	0.76	-1.04	1.86	1.81	-0.65	-0.45	0.30	-0.40	-0.65	0.003	0.001	-0.001	-0.002	-0.003	0
4.800	0.76	-1.04	1.86	1.81	-0.65	-0.45	0.29	-0.40	-0.65	0.003	0.001	-0.001	-0.002	-0.003	0
4.900	0.76	-1.04	1.86	1.81	-0.65	-0.45	0.29	-0.40	-0.65	0.003	0.001	-0.001	-0.002	-0.003	0
5.000	0.75	-1.04	1.86	1.81	-0.65	-0.45	0.29	-0.40	-0.65	0.003	0.001	-0.001	-0.001	-0.002	0
5.100	0.75	-1.04	1.86	1.81	-0.65	-0.45	0.29	-0.40	-0.65	0.003	0.001	-0.001	-0.001	-0.002	0
5.200	0.75	-1.04	1.86	1.81	-0.65	-0.45	0.29	-0.40	-0.65	0.003	0.001	-0.001	-0.001	-0.002	0
5.300	0.75	-1.04	1.86	1.81	-0.65	-0.45	0.29	-0.40	-0.65	0.002	0.001	-0.001	-0.001	-0.002	0
5.400	0.75	-1.04	1.86	1.81	-0.65	-0.45	0.29	-0.40	-0.65	0.002	0.001	-0.001	-0.001	-0.002	0
5.500	0.75	-1.04	1.86	1.81	-0.65	-0.45	0.29	-0.40	-0.65	0.002	0.001	-0.001	-0.001	-0.002	0
5.600	0.75	-1.04	1.86	1.81	-0.65	-0.46	0.29	-0.40	-0.65	0.002	0.001	-0.001	-0.001	-0.002	0
5.700	0.75	-1.04	1.86	1.81	-0.65	-0.46	0.29	-0.40	-0.65	0.002	0.001	-0.001	-0.001	-0.002	0
5.800	0.75	-1.04	1.86	1.81	-0.65	-0.46	0.29	-0.40	-0.65	0.002	0.001	-0.001	-0.001	-0.002	0

// JOB 000F CCLCA 000B COOB COOA

LOG DRIVE	CART SPEC	CART AVAIL	PHY DRIVE
0000	000F	000F	0001
0001	000A	000A	0002
0002	000F	000B	0000

V2 P11 ACTUAL 16K COME 16K

// CLP

*DFILE LA CJO3 8 000B
D 06 ENTRY POINT NAME ALREADY IN LET/FLET

ORIGINAL PAGE IS
OF POOR QUALITY

PAGE 1 10/18/71
// JOB GOOF CCLA GOCH COOR GOOA

LOG DRIVE CART SPEC CART AVAIL PHY DRIVE
CCCC CCGF CUGF CCG1
CCG1 CCGA CCGA CCG2
CCG2 CCGF CCGR CCGO

V2 #11 ACTUAL 16K CONFIG 16K

// FORTRAN
*IOCS(DISK)

C-ERRS...STNO.C..... FORTRAN SOURCE STATEMENTS IDENTFCN **COMPILER MESSAGES**

REAL K(101,5,2),7(101)
DATA NPIS/101/
DEFINE FILE 1(112,12,0,100MY)

C
C
C

DO 1 N=1,NPTS
READ(1,N) TIN,((K(N,I,J),I=1,5),J=1,2)
1. CONTINUE

C
C

PAUSE
CALL PSTZE(5.0,4.0)
CALL PLNGC(1.0,6.0)
CALL PRGX
CALL PAXS
DO 2 I=1,5
CALL PLCH(0.0,T,10.0,-2.0,K(1,I,1),2,C,NPTS)

2. CONTINUE
CALL PLNGC(0.0,-5.0)
CALL PRGX
CALL PAXS
DO 3 I=1,5
CALL PLCH(0.0,T,10.0,-2.0,K(1,I,2),2,C,NPTS)
3. CONTINUE
CALL PLNGC(-1.0,-1.0)
CALL EXIT
END

FEATURES SUPPORTED
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CORE REQUIREMENTS FOR -
COMPCH- C, VARIABLES AND TEMPORARIES- 2242, CONSTANTS AND PROGRAM- 218

END OF SUCCESSFUL COMPIATION

// XEC 1

*FILLS(1,0003)

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